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A Differential Analyzer for the Schrödinger Equation*

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A rapid differential analyzer of accuracy about 0.5 percent is discussed and its performance reviewed. Extensions to differential equations other than that of Schrödinger are mentioned.

INTRODUCTION

FOR some time there has been a need for a small, simple differential analyzer. For example, in nuclear physics it is most convenient to have available a rapid method of solution of the one-dimensional Schrödinger equation,

$$(\partial^2\varphi/\partial x^2)+[E-V(x)]\alpha\varphi=0, \qquad (1)$$

for a large variety of functions $V(x)$. The problem undertaken in this research was primarily to design and build a machine capable of solving this equation, and, secondarily, to develop methods to be used in extending the usefulness of the machine to other problems. For most purposes an accuracy of solution of the order of one percent is sufficient, but 0.1 percent is desirable, and, as will be seen, readily achieved.

METHOD OF ATTACK

There are, in general, at least two overlapping classes of computing machines; analog and digital computers. In the Schrödinger problem either type may be used; for instance, a moving needle galvanometer with zero friction and zero restoring torque, the axis of whose coil coincides with the axis of the magnetized needle in the zero position, obeys the Schrödinger equation in the sense that if the current through the coil is made to vary in proportion to $[E-V(t)]$, the angular position of the needle as a function of time will be the solution of the equation

$$(\partial^2\theta/\partial t^2)+(K/I)[E-V(t)]\theta=0$$

over a considerable range of θ. This analog of the Schrödinger particle was not employed as a computer because of its severely limited scope.

A general analog differential analyzer (Bush type) adapted to the problem at hand has the form schematized in Fig. 1. Here a, b, c, and d are variables of the physical form required for the operation of the machine, i.e., shaft rotations, voltages, etc., and b_0, c_0 the initially given or assumed values of b, c. Now, from the nature of the individual units

$$a=db/dx, \quad b=dc/dx; \quad a=d^2c/dx^2$$
$$d=[E-V(x)]\alpha c\equiv a;$$

so:

$$(d^2c/dx^2)+[E-V(x)]\alpha c=0, \qquad (2)$$

which differs from (1) only by a simple change of variable.

In a certain sense, a digital computer is the ideal type for almost any problem, especially if adapted to take data in the form of tables, graphs, or even rough sketches, but in the present state of the subject a small digital computer, though a powerful tool, is not available for this purpose. It should be noted, however, that if, as seems likely, storage mechanisms are improved, a small immensely flexible digital computer would certainly be a possibility.

It was desired to perform the required operations electrically insofar as possible to dispense with the usual gears, shafts, servos, etc. With the electrical methods available, however, a great simplification is achieved by making the independent variable the time, (t). Electrical methods have the advantage of flexibility, accuracy, economy, but suffer from limitations, the most severe of which restricts the most straightforward machine to linear differential equations.

The heart of the analyzer is the integrator unit which is derived from the simple resistance-capacitance integrator of Fig. 2 as follows.

The voltages in this circuit are connected by the relation $dE_0/dt=(Ei-E_0)/\tau$ where $\tau=RC$. Considering the voltages as variables, this network would be a perfect integrator if the term E_0/τ could be removed. Physically this term arises because the junction of the resistor and condenser is not always at ground potential, or in other words, τ is non-infinite. Evidently τ itself cannot be made infinite without losing a large factor in voltage in each integrator, so another method is employed; namely, to keep this junction at ground poten-

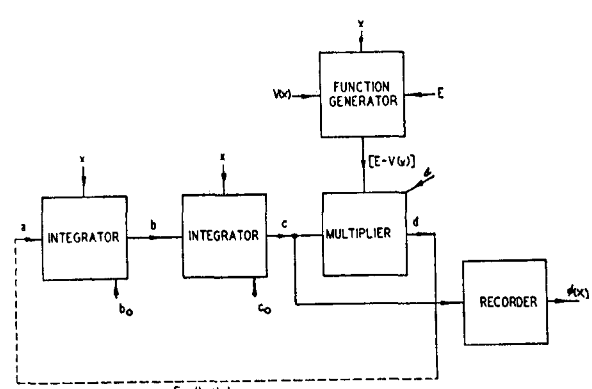

FIG. 1. Bush type Schrödinger differential analyzer.

* Part of this work was performed under an AEC Predoctoral Fellowship.

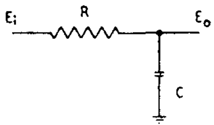

FIG. 2. Prototype electrical integrator.

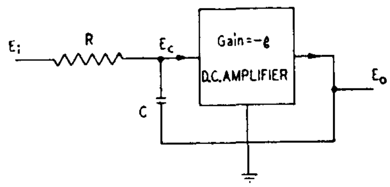

FIG. 3. Basic compensated integrator.

tial by using a feed-back amplifier to vary the potential of the bottom terminal of the condenser (Fig. 3).

Now the continuity equations for current give

$$\frac{E_i - E_c}{R} = \frac{-d(E_0 - E_c)C}{dt} = -\frac{d}{dt}\left(\frac{g+1}{g}CE_0\right)$$

and for g large (in our case at least 10^6)

$$\frac{-dE_0}{dt} = \frac{E_i}{RC} + \frac{E_0/g}{RC} \approx \frac{E_i}{RC}.$$

It is to be noted that accuracy of integration depends only on the minimum variational g during the integration period, as well as on unwanted components of E_i, τ.

The major difficulties were three. It was found that most resistors of the value necessary are violently nonlinear, but certain sputtered film or deposited carbon resistors afford a convenient solution to the problem.[1] The restriction is really quite stringent since for 0.1 percent accuracy in each integrator and 200 v applied to the resistors, a voltage coefficient of resistance less than 10^{-5}/v is required.

The second problem was non-ideal behavior of the condenser. Most condensers are satisfactory as regards leakage (it is evidently only terminal-to-terminal leakage that counts, since leakage to ground enters only as it affects g), but it is difficult to find a condenser which does not possess some dielectric "soak-in." A quite satisfactory solution to the difficulty was found in polyethylene film condensers[2] with leakage time constants $>10^6$ sec. and dielectric soak-in about 10^{-4}.

The third and by far the most difficult problem was the input behavior of the d.c. amplifier, that is, grid current and grid-cathode potential changes. It was felt that the use of standard receiving tube types was an important advantage worth considerable effort to achieve. Grid current was finally reduced to less than 10^{-9} amp. without tube selection by using a 6AK5 at 100-μa plate current. The grid-cathode potential was stabilized by regulating the average heater power to better than 0.1 percent by the scheme[3] illustrated in Fig. 4. In this circuit the filament of the first diode is operated at such current that the emission is temperature-limited. It is then easily shown that the sensitivity of the space current to heater voltage variation gives about 2 v change in plate potential for 0.1 percent heater voltage change. This error signal is then amplified and applied through the power output tube and transformer so as to maintain constant heater voltage.

After testing many circuit designs, several of which operated with considerable success (cascodes), the amplifier sketched in Fig. 5 was standardized. The amplifier pictured is connected as an integrator by the feed-back

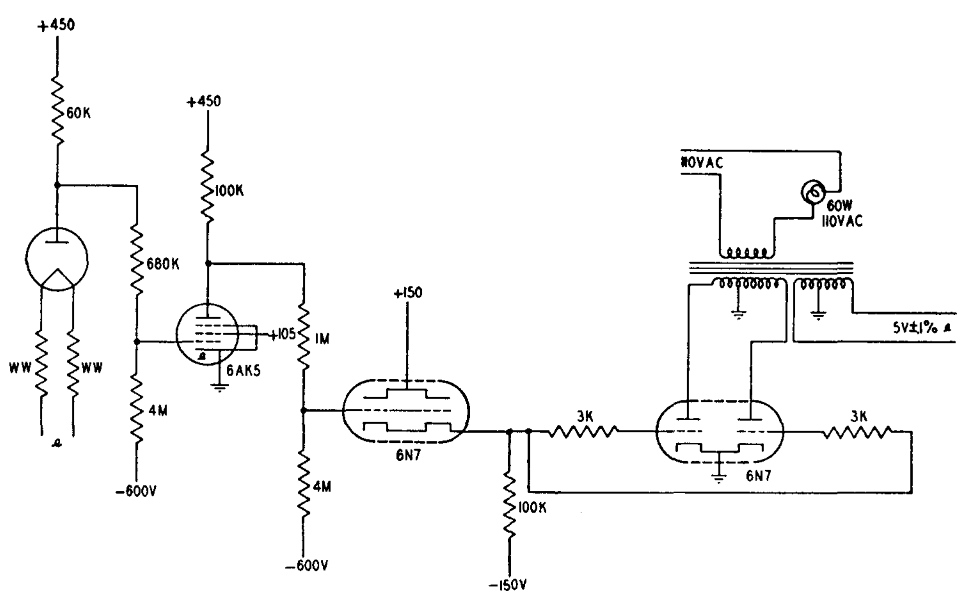

FIG. 4. An a.c. r.m.s. voltage regulator.

[1] *IRC* type *DCH*, linear to 10 parts/million/volt.
[2] Condenser Products Company, Chicago, Illinois, Type *LAC*.
[3] Independently developed, but similar to that mentioned by M. M. Hubbard and P. C. Jacobs, *Radar System Engineering* (McGraw-Hill Book Company, Inc., New York, 1947), p. 564.

condenser C. An important point to note is that because the non-fed-back gain of the amplifier is greater than 10^6, the grid of the first tube is always within 0.2 mv of ground, for an output voltage of -150 to $+150$ v. This fact allows the use of a simple method of setting initial conditions for integrators: If the input is grounded and the switch S_2 is closed, the output potential of the amplifier must adjust itself so that the first grid potential is zero. This can only be done by seeking a well-defined output potential for each setting of the coarse and fine initial condition potentiometers in the lower left-hand corner. (The initial condition is, of course, set by observing the output potential with the machine's accurate servomechanical d.c. voltmeter and adjusting the potentiometers to give the desired voltage.)

On opening S_2 no potential change larger than 0.1 mv is expected or observed. The switch must, however, be so modified by filing that the grid contact "makes" last and "breaks" first, for obvious reasons.

The properties of the amplifier, in view of its practically infinite gain, can be characterized by the behavior of the input circuit: grid current is always negligible, while grid-cathode potential changes (which may be compensated by varying the $5K$ cathode resistor) are less than 1 mv/hr. after a few minutes warm-up. It may be remarked that a grid potential change of 10 mv would be produced by a leakage resistance to the grid of 5×10^{11} ohms from a 500-v source. The trick is that the grid circuit is entirely separated from points at different potential by a grounded "guard ring," i.e., on S_2 only half the contacts are used, the others being grounded, while all grid wiring is enclosed in shielded cable. Thus the problem is simplified to that of keeping grid-to-ground leakage above 10^6 ohm, which is readily done. The only adjustment needed or possible in the amplifier is the compensation of grid-cathode potential changes by means of the variable first cathode resistor. This is accomplished by setting the input switch to $1M$ with input grounded. Any grid-cathode unbalance E then produces a rate of rise of the output voltage E v/sec. (for a 1-mf condenser) which is to be reduced to zero by varying the cathode resistor. The lack of grid current may be checked by observing that zero drift rate on $1M$ means also zero drift with the input switch at $10M$. Drift rates at high output voltages were checked by observing the output voltage with a 50-mv meter and a bucking battery. This drift rate is equal to that at zero output voltage because of the high amplifier gain.

For the usual integrating period of 30 sec., the amplifier characteristics are such that drift errors are held to 0.1 percent of full scale even on the shortest

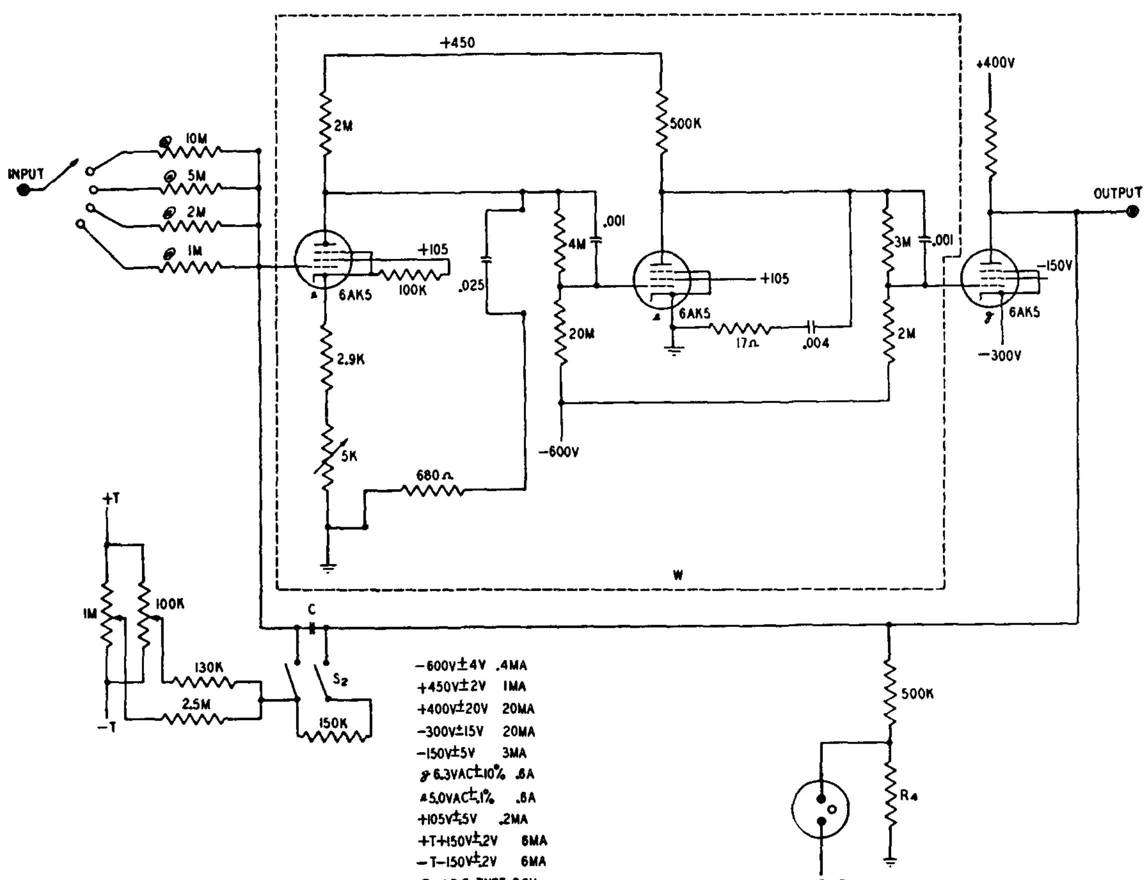

FIG. 5. Standardized d.c. feed-back amplifier.

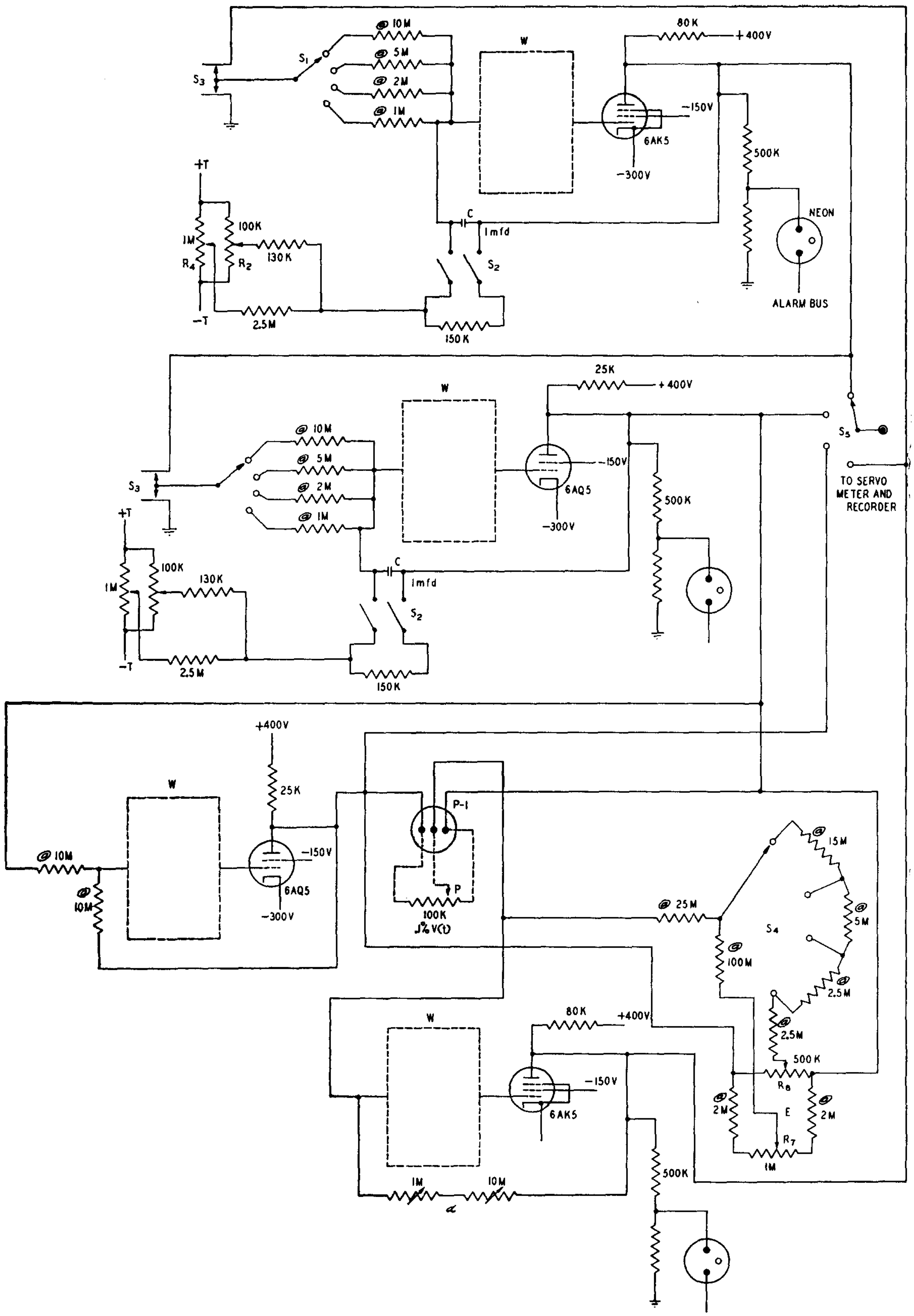

FIG. 6. Computing section.

time constant. The networks between first and second and between the second and third stages serve to stabilize the amplifier against high frequency oscillations, while low impedance (non-regulated) and separate power supplies for the first two and last stages serve to prevent low frequency instability. A power supply yielding many voltages was chosen rather than a more complicated integrator since the power supply may be used for a much larger differential analyzer without undue current requirements. With the proper output tube (6AQ5) and plate resistor, the amplifier is capable of an output voltage range of plus 150 to minus 150 v into a 10,000-ohm load, with, of course, negligible effective output impedance. Relays between units maintain isolation until the desired instant, and an overload signal is available from the neon lamp. The same amplifier may be used with a feed-back resistor in place of the integrating condenser to give an inverting amplifier, impedance changer, etc.

The overloading of any amplifier automatically turns off the multivibrator-controlled relays S_3 (Fig. 6), thus allowing the operator to adjust the various multiplying factors and initial conditions so as to keep the variables within range. This multivibrator is operable also from a start-stop foot switch and from contacts bearing on pencil lines marked on the graph carrying the potential function, allowing precise control of the integrating period. The scheme of the entire machine except for power supply and servos is shown in Fig. 6. Not shown in Fig. 6 are 20-mmf condensers across the feed-back resistors in the amplifier and multiplier sections. These, together with parallel combinations of 2200-ohm resistors and 0.01-mf condensers in the cathode of each of the four output tubes, effectively stabilize the system against phase shift oscillations caused by the capacitance to ground in shielded cables.

The machine thus consists of four "package" amplifiers identical except that two have polyethylene condensers as the feed-back elements, while two have linear resistances. The purpose of the integrators is seen from Fig. 1. The third unit is used as a phase inverter to give a voltage equal to and of polarity opposite from that appearing at the output of the second integrator. These two equal and opposite voltages are then added in varying proportion by means of the linear potentiometer P and the stable potentiometers R_6 and R_7. Thus if y is the voltage appearing at the output of the second integrator, $-y$ appears at the output of the phase inverter and $[E-V(x)]y$ at the input to the fourth unit. This fourth amplifier is really an impedance changer, its output completing the differential analyzer feed-back loop by being connected to the input of the first integrator.

During operation the finger of the linear potentiometer (P) is constrained, by hand or with a photo-cell +servo system, to follow the graph of $V(X)$ carried on a rotating drum, while another linear potentiometer and servo record the solution $\varphi(t)$ on a synchronously rotating cylinder.

PERFORMANCE

In order to find an eigenvalue and eigenfunction of a given Schrödinger potential, a trial E is chosen and a solution computed on the machine, with convenient initial conditions. By node counting and interpolation, a better trial value is found and the process repeated until the desired accuracy is achieved. Usually five repetitions (30 sec. per integration, 10 min. total) suffice for 0.1 percent accuracy in E.

Three identical brass scales are used with the machine: one on the servo output meter (equipped also with a switch increasing the sensitivity ten times), one on the pointer connected to the linear potentiometer P, and one for use in drawing input potential functions. The first step in the solution is to draw the potential function as large as will allow it to fit on the input graph, and to any convenient abscissa $x(t)$.

Next, one has to adjust the individual sections of the machine. This is done by throwing relays S_3 into the "rest" position by means of the foot switch and varying the cathode resistor in the first integrator until no drift is apparent. Evidently the switch S_1 should be set at $1M$ for this purpose. The same is done for the second integrator. Now with the output of the second integrator at zero potential the cathode resistor in the inverter is adjusted to give zero output voltage. With the output of the second integrator (and thus the inverter output) held at zero, the cathode resistor of the fourth unit is now adjusted until its output is zero. The only remaining adjustment consists of choosing the free oscillation period of the machine so as to make the analog "mass" of the analyzer equal to that of the Schrödinger particle. The mass of the particle is entirely fixed by wave-length at some energy E_0. The input potentiometer P is thus set at E_0 on the graph and the switches S_1 adjusted to give approximately the right period. The machine is then allowed to make an oscillation at this frequency, and the wave-length is measured. Any discrepancy between the Schrödinger wave-length and the period of the machine is then removed by changing the multiplying factor α in proportion to the square of the ratio of the wave-lengths. This also can be done without calibrated dials by using the servo voltmeter.

After this has been accomplished the physical analogy is complete and one can search for eigenvalues without the necessity of readjustment except at intervals of some hours.

A very rough estimate of the desired eigenvalue is made. The machine is then allowed to perform its solution of the differential equation for this trial eigenvalue. If, for instance, the ground state is desired, one counts the number of nodes in the trial solution. If the trial solution has a node a lower trial eigenvalue is selected and the solution drawn by the machine. By a simple process of extrapolation a better trial eigenvalue is found, two more trial solutions usually being sufficient to give 0.1 percent sensitivity in the eigenvalue.

The machine may be relied upon, when used properly, to give solutions of absolute accuracy 0.2 percent in not unfavorable cases. Of course, the machine solution must often be supplemented by asymptotic analytic solutions, especially at infinities in the potential.

This differential analyzer is also immediately suited to the solution of the paraxial equations for the path of charged particles in cylindrically symmetric magnetic fields and affords a considerable saving in time. Indeed, any second-order differential equation $y''+X_1y'+X_2y=0$ may be solved on this machine after first making the substitution $y=u\exp(-\frac{1}{2}\int X_1dx)$ to give $u''+[X_2-\frac{1}{2}X_1-\frac{1}{4}X_1^2]u=0$.

It is hoped to use the analyzer for the computation of nuclear potentials for nuclear shells. By adding two more input graphs the present model may be adapted to the solution of the relativistic Dirac equation where its main purpose may well be to provide experience and intuition as to the behavior of the solutions.

By the use of magnetic clutches[4] or hydraulic servo mechanisms operating on control powers of only a watt or so, it should be possible to perform multiplication, and hence all processes necessary to solve non-linear differential equations with linear circuits and potentiometers. Because of the use of linear elements, these solutions should be accurate to perhaps 0.1 percent with care.

ACKNOWLEDGMENT

The writer takes this opportunity to thank Professor E. Fermi for much help and encouragement in the prosecution of this project.

[4] Similar to those described by E. S. Bettis and E. R. Mann, Rev. Sci. Inst. 20, 97 (1949).

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Design and Operation of Liquid Nitrogen-Cooled Solenoid Magnets*

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This paper describes the design and construction of compact iron-free solenoid magnets cooled by immersion in liquid nitrogen. Construction details and performance data are given for two such magnets, which produce a magnetic field of 4000 gauss with a power expenditure of 2.7 to 3.3 kw. Heat transfer coefficients between copper wire and boiling liquid nitrogen have been measured for temperature differences between the wire and the liquid of 2 to 11°K.

FOR many types of magnetic measurements, it is desirable to obtain large magnetic fields by use of an iron-free solenoid. This is particularly true for measurements at the temperatures of liquid helium, where differential magnetic susceptibilities are often required. If such measurements are to be made precisely, hysteresis in the magnet must be eliminated, and the interaction between the magnet and small measuring fields must be reduced to a minimum. These conditions are best met by the use of a magnet which contains no ferromagnetic material.

TABLE I. Design requirements for liquid nitrogen-cooled magnets.

Magnetic field at full power	4000 to 5000 gauss
Variation of magnetic field along axis	No more than one percent over 1 cm from center
Maximum power expenditure	4 kw
Maximum consumption of liquid nitrogen	1200 cm³/min.
Available voltage	190 v
Maximum current (for convenience)	30 amp.
Inside diameter	3 in.
Maximum outside diameter	6 in.

In designing a solenoid magnet, it is desirable to obtain as large a magnetic field as possible for a given amount of power under given conditions of working space and homogeneity. To achieve this, the magnet should be constructed from wire of as low resistivity as possible. It should be as compact in size as other conditions permit, and the space enclosed by the windings should be filled, insofar as possible, by the wire itself. Adequate provisions must be made for electrical insulation and for the removal of heat produced during operation. Wire size, total resistance, and the nature of electrical connections depend mainly upon the current or voltage characteristics desired, and do not greatly affect the power expenditure.

For most installations, these conditions can be met by a multilayer winding of relatively large wire, preferably of square or rectangular cross section, with spacings between turns and layers smaller than the wire size. Cooling can be accomplished by circulation of a non-polar liquid of low viscosity.

* This work was supported in part by the ONR under contract with The Ohio State University Research Foundation.
[1] Now at Pennsylvania State College.